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Some remarks on Ádám's conjecture for simple directed graphs

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Abstract

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The classes of multidigraphs for which Ádám's conjecture (that any digraph containing a directed cycle has an arc whose reversal decreases the total number of directed cycles) does not hold, were described in Grinberg (1988), Jirásek (1987) and Thomassen (1987). The question remains, however, open for simple directed graphs. In the paper we show that the conjecture holds for all simple digraphs containing a nontrivial strongly connected component which is not strongly 2-connected and for simple digraphs that become acyclic after reversal of at most three of their arcs.

In [1] A. Ádám formulated the following conjecture.

> For every digraph $G = (V, A)$ containing a directed cycle there is an arc $\langle x, y \rangle \in A$ whose reversal decreases the total number of its cycles.

The conjecture can also be formulated by means of the number of directed paths. Let (x, y) denote the number of the paths from the vertex x to the vertex y. Then the number of the cycles containing the arc $\langle x, y \rangle$ is just (y, x) and after its reversal there will be $(x, y) - 1$ cycles containing this reversed arc. Hence Ádám's conjecture can be formulated as follows:

> For every non-cycle-free digraph G there is an arc $\langle x, y \rangle$ such that $(x, y) - 1 < (y, x)$.

Now, it can be easily seen that the following propositions hold.

Proposition 1. *Ádám's conjecture holds for every digraph containing a symmetric pair of arcs.*

Proof. If $\langle x, y\rangle$ and $\langle y, x\rangle$ are symmetric arcs of a digraph, then either $(x, y) \leqslant (y, x)$ or $(y, x) \leqslant (x, y)$ holds and so, reversal of one of these arcs decreases the total number of cycles. □

Proposition 2. *If in a cut of a connected digraph there exists exactly one arc $\langle x, y\rangle$ having the opposite direction than the other arcs of the cut and there is a cycle containing the arc, then reversal of this arc decreases the total number of cycles of the digraph.*

Proof. The arc $\langle x, y\rangle$ is the only one having the opposite direction than the others, so $(x, y) = 1$. Since there is a cycle containing $\langle x, y\rangle$, $(y, x) \geqslant 1$ holds. So $(x, y) \leqslant (y, x)$, and hence the theorem. □

Proposition 3. *If a digraph G contains a directed path of length 2 whose reversal decreases the number of the cycles of G then reversal of one of these two arcs also decreases the number of its cycles.*

Proof. Let $\langle x, y\rangle$ and $\langle y, z\rangle$ be the arcs of the path. Suppose reversal none of them decreases the total number of cycles. Then $(x, y) - 1 \geqslant (y, x)$ and $(y, z) - 1 \geqslant (z, y)$ hold. Since reversal of the path produces at least $(x, y) - 1 + (y, z) - 1$ new cycles and at most $(y, x) + (z, y)$ cycles of G are destroyed, the previous inequalities imply that reversal of the path does not decrease the total number of cycles. This is a contradiction. □

Proposition 4. *If a digraph G contains a directed cycle of length 3 whose reversal decreases the total number of its cycles then reversal of one of these three arcs also decreases the number of its cycles.*

Proof. Let $\langle x, y\rangle$, $\langle y, z\rangle$ and $\langle z, x\rangle$ be the arcs of the cycle. Note that any cycle produced by reversal of one of these three arcs contain none of the others. So, at least $(x, y) - 1 + (y, z) - 1 + (z, x) - 1$ new cycles are produced by reversal of this triangle. Besides, the new triangle $\langle x, z\rangle$, $\langle z, y\rangle$, $\langle y, x\rangle$ is produced. In the same time each cycle of G containing at least one of these three arcs is destroyed. The number of such cycles is not greater than $(y, x) + (z, y) + (x, z) + 1$ (there is just one cycle in G containing all these three arcs).

Now, suppose reversal none of these arcs decreases the total number of cycles in G. Then $(x, y) - 1 \geqslant (y, x)$, $(y, z) - 1 \geqslant (z, y)$ and $(z, x) - 1 \geqslant (x, z)$ hold. By summation we obtain

$$(x, y) - 1 + (y, z) - 1 + (z, x) - 1 + 1 \geqslant (y, x) + (z, y) + (x, z) + 1$$

and so, reversal of all three arcs does not decreases the total number of cycles in G. This is a contradiction. □

The propositions proved above hold for any digraphs (without loops). Some infinite classes of multidigraphs (without symmetric arcs) for which Ádám's conjecture does not hold were described in [3–5]. The question remains, however, open for simple digraphs.

Let us call a simple digraph G strongly 2-connected if in each of its cuts there exist at least 2 arcs in one direction and at least 2 arcs in the opposite direction. Then the class of simple digraphs for which Ádám's conjecture holds can be described as follows.

Theorem 1. *Ádám's conjecture holds for every simple digraph containing a nontrivial strongly connected component which is not strongly* 2*-connected.*

Proof. Consider the component in the theorem. Because of its strong connectivity each of its arcs is contained in a cycle and every cut of the component contains at least one arc in both directions. In the same time there must be a cut, containing in one of the directions just one arc. So according to Proposition 2 the theorem follows. □

Remark 1. According to Theorem 1, Ádám's conjecture holds for every digraph containing a strongly connected component with minimal indegree (resp. outdegree) at most 1 (e.g. outerplanar).

Remark 2. The counterexamples to Ádám's conjecture found in [3–5] were multidigraphs of the type G^p, which can be constructed from a simple digraph G by replacing each arc by p parallel ones. It is not difficult to see that if G fulfils the assumptions of Theorem 1, the Ádám's conjecture also holds for all multidigraphs G^p.

Now, it will be shown that Ádám's conjecture holds for simple digraphs that become acyclic after reversal of several arcs.

Theorem 2. *If after reversal of at most three arcs a non-cycle-free simple digraph G becomes acyclic then there exists an arc in G whose reversal decreases the total number of its cycles.*

Proof. In the acyclic digraph obtained from G after arc reversal there is at least one source vertex s with indegree 0 and at least one sink t $(t \neq s)$ with outdegree 0. To obtain this, all arcs ending at s and all arcs beginning at t in G must be reversed. If G becomes acyclic after reversal of at most three arcs then it contains at most one nontrivial strongly connected component which is strongly 2-connected. In the case that it does not contain any such component according to Theorem 1, Ádám's conjecture holds for it. In the other case without loss of generality we can assume that G is strongly 2-connected.

Hence each vertex of G has indegree and outdegree at least 2 and so three arcs must be reversed to make G acyclic. Further, in the obtained acyclic digraph there is just one source s and just one sink t. Indegree of s and outdegree of t in G have to be 2, the arc $\langle t, s\rangle$ has to be in G and this arc together with arcs $\langle t, x\rangle$ and $\langle y, s\rangle$ had to be reversed.

Let us linearly order the vertices of the obtained acyclic digraph in such a manner that the occurrence of a directed path from a to b in the digraph implies $a<b$ (s is the first and t is the last one in this ordering). In the case that $y<x$ let us consider the cut (V_1, V_2) of G, where in V_1 there are all vertices not greater than y and the others are in V_2. Then, because of the property of the ordering described above, the only arc from V_2 to V_1 is $\langle t, s\rangle$ which is in contradiction with strongly 2-connectivity of G. So, $x \leqslant y$ must hold. We show, that reversal of the arc $\langle t, s\rangle$ decreases the total number of cycles. The number of the new cycles produced by reversal of $\langle t, s\rangle$ is equal to the number of all dipaths from t to s except the arc $\langle t, s\rangle$. Each such dipath starts with $\langle t, x\rangle$ and ends with $\langle y, s\rangle$. So, the number of such dipaths is equal to the number of all dipaths from x to y not containing the arc $\langle t, s\rangle$ (if $x=y$ then it is equal to 1).

At the same time all cycles in G containing the arc $\langle t, s\rangle$ are destroyed by reversal of $\langle t, s\rangle$. The number of such cycles is equal to the number of all dipaths from s to t. Since there are dipaths

$$yy_1 \cdots y_m t, \quad \text{where } m \geqslant 0, \quad y<y_1<\cdots<y_m<t$$

and

$$sx_n \cdots x_1 x, \quad \text{where } n \geqslant 0, \quad s<x_n<\cdots<x_1<x$$

(because each vertex in G has indegree and outdegree at least 2; see Fig. 1), the number of the cycles destroyed by reversal of $\langle t, s\rangle$ is not smaller than the number of dipaths from x to y not containing $\langle t, s\rangle$ (which is equal to the number of new cycles). To show that it is greater it is sufficient to see that there must exist

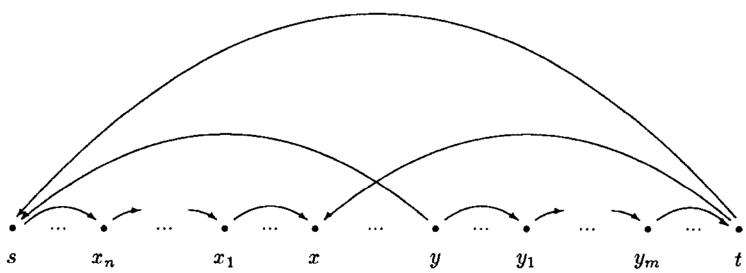

Fig. 1.

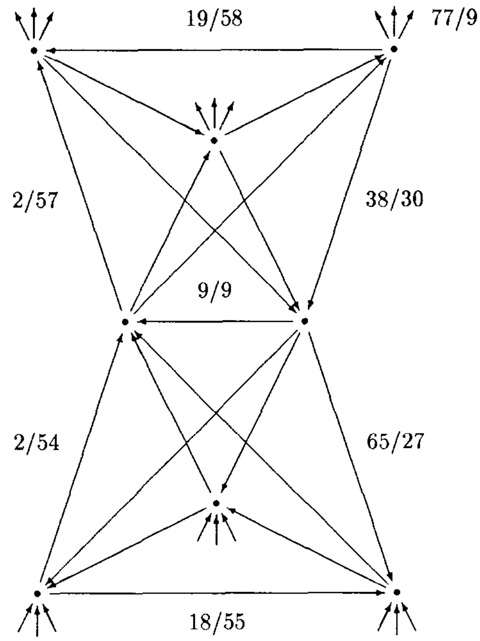

Fig. 2.

an dipath $sz_1 \cdots z_k t$, where $s < z_1 < \cdots < z_k < t$ and $z_1 \neq x_n$ (for the same reason as above). So, the theorem follows. □

Remark 3. Let us call a pseudocycle to an arc $\langle x, y \rangle$ any path from x to y different from the arc $\langle x, y \rangle$ together with the arc $\langle x, y \rangle$. If reversal of an arc $\langle x, y \rangle$ does not decrease the number of cycles in G then there is at least so many pseudocycles to the arc $\langle x, y \rangle$ as the cycles containing $\langle x, y \rangle$. If G does not satisfy Ádám's conjecture then the property formulated above holds for each of its arcs. This means that the sum of the lengths of all cycles in G is not greater than the number of all pseudocycles in G. However, this is not a sufficient condition as the following example shows. In the digraph at Fig. 2 each vertex of the triangle at the top should be connected with each vertex of the bottom triangle. Eight of its arcs are labeled by the number of pseudocycles to the arc (i.e., $(x, y) - 1$ for the arc $\langle x, y \rangle$) and the number of cycles containing the arc (i.e., (y, x) for the arc $\langle x, y \rangle$). The labels of the other arcs can be obtained using the symmetry of the digraph. The digraph has 1134 pseudocycles while the sum of lengths of all its cycles is 933 but it fulfils Ádám's conjecture.

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